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A History-Dependent Random Sequence Defined by Ulam*

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Ulam has defined a history-dependent random sequence of integers $\{a_n\}$ by the recursion $a_{n+1} = a_n + a_{X(n)}$, $a_1 = 1$, $P\{X(n) = K\} = n^{-1}$, and $K = 1, 2, \ldots, n$. It is shown that the expectation of a_n is asymptotic to $\exp(2\sqrt{n})$ and that the expectation of a_n^2 is asymptotic to $\exp[\sqrt{2[5+\sqrt{17}]}\sqrt{n}]$. The methods of generating functions and steepest descent are used. © 1989 Academic Press, Inc.

1. Let $X(n)$ be independent random variables such that

$$P\{X(n) = K\} = 1/n, \qquad K = 1, 2\ ,$$

and let

$$a_{n+1} = a_n + a_{X(n)}, \qquad a_1 = 1.$$

We have

$$m_{n+1} = E(a_{n+1}) = m_n + \frac{1}{n}\sum_{K=1}^{n} m_K,$$

or

$$nm_{n+1} = nm_n + \sum_{K=1}^{n} m_K. \tag{1}$$

*This note is a faithful reproduction of the unclassified Los Alamos Report LA-4289, UC-32, Mathematics and Computers TID-4500. It was written in October 1969 and distributed in March 1970. Although it does not appear that the author ever intended to publish it in a journal (he died in 1985), the editors think that the readers will appreciate this short display of Kac's computational virtuosity.

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Let

$$f(z) = \sum_{n=1}^{\infty} m_n z^{n-1} = m_1 + m_2 z + m_3 z^2 + \cdots,$$

and note that Eq. (1) gives

$$f'(z) = \frac{d}{dz}(zf(z)) + \frac{1}{1-z}f(z) = zf' + f + \frac{1}{1-z}f.$$

Thus,

$$(1-z)f' = \left(1 + \frac{1}{1-z}\right)f$$

and

$$f' = \left(\frac{1}{1-z} + \frac{1}{(1-z)^2}\right)f,$$

and hence (using the initial condition a_1 $(= m_1) = 1$),

$$f(z) = \frac{1}{1-z}e^{z/(1-z)} = \sum_{n=0}^{\infty} L_n^{(1)}(1) z^n, \tag{2}$$

where $L_n^{(1)}(t)$ are familiar Laguerre polynomials. Elementary application of the method of steepest descent gives

$$\boxed{\log m_n \sim 2\sqrt{n}\,.}$$

2. One can also calculate the second moment. We have

$$a_{n+1} = a_n + a_{X(n)}$$

and

$$a_{n+1}^2 = a_n^2 + a_{X(n)}^2 + 2a_n a_{X(n)},$$

therefore,

$$\begin{aligned}\sigma_{n+1} = E\{a_{n+1}^2\} &= \sigma_n + \frac{1}{n}\sum_{K=1}^{n}\sigma_K + \frac{2}{n}\sum_{K=1}^{n} E\{a_n a_K\} \\ &= \sigma_n + \frac{1}{n}\sum_{K=1}^{n}\sigma_K + \frac{2}{n}\sigma_n + \frac{2}{n}\sum_{K=1}^{n-1} E\{a_n a_K\}.\end{aligned}$$

Now set

$$\alpha_n = \sum_{K=1}^{n-1} E\{a_n a_K\}, \qquad \alpha_1 = 0, \qquad \alpha_2 = 2,$$

and note that

$$\begin{aligned}
\alpha_{n+1} &= \sum_{K=1}^{n} E\{a_{n+1} a_K\} \\
&= \sum_{K=1}^{n} E\{(a_n + a_{X(n)}) a_K\} = \sum_{K=1}^{n} E\{a_n a_K\} + \sum_{K=1}^{n} E\{a_{X(n)} a_K\} \\
&= \sigma_n + \alpha_n + \frac{1}{N} \sum_{K=1}^{n} \sum_{l=1}^{n} E\{a_l a_K\} \\
&= \sigma_n + \alpha_n + \frac{1}{n} \sum_{K=1}^{n} \sigma_K + \frac{2}{n} \sum_{1 \le K < l \le n} E\{a_K a_l\} \\
&= \sigma_n + \frac{1}{n} \sum_{K=1}^{n} \sigma_K + \alpha_n + \frac{2}{n} \sum_{l=1}^{n} \alpha_l .
\end{aligned}$$

A subtle point is involved here. What we use is the chain of equations:

$$\begin{aligned}
E\{a_{X(n)} a_K\} &= \sum_{l=1}^{n} \frac{1}{n} E\{a_{X(n)} a_K | X(n) = l\} \\
&= \frac{1}{n} \sum_{l=1}^{n} E\{a_l a_K\}.
\end{aligned}$$

But this is true only if $K \le n$. To see what is involved, it is enough to consider

$$E\{a_{X(n)} a_K | X(n) = l\},$$

which, for $K \le n$, is indeed $E\{a_l a_K\}$. However, suppose that $K = n + 1$, then

$$\begin{aligned}
E\{a_{X(n)} a_{n+1} | X(n) = l\} &= E\{a_{X(n)} (a_n + a_{X(n)}) | X(n) = l\} \\
&= E\{a_l a_n\} + E\{a_l^2\},
\end{aligned}$$

while ($l \le n$),

$$\begin{aligned}
E\{a_l a_{n+1}\} &= E\{a_l a_n\} + E\{a_l a_{X(n)}\} \\
&= E\{a_l a_n\} + \frac{1}{n} \sum_{m=1}^{n} E\{a_l a_m\}.
\end{aligned}$$

In summary,

$$\boxed{\begin{aligned} \sigma_{n+1} &= \sigma_n + \frac{1}{n}\sum_{K=1}^{n}\sigma_K + \frac{2}{n}\sigma_n + \frac{2}{n}\alpha_n,\\ \alpha_{n+1} &= \sigma_n + \frac{1}{n}\sum_{K=1}^{n}\sigma_K + \alpha_n + \frac{2}{n}\sum_{K=1}^{n}\alpha_k. \end{aligned}}$$

Set

$$g(z) = \sum_{n=1}^{\infty}\sigma_n z^{n-1}$$

and

$$h(z) = \sum_{n=1}^{\infty}\alpha_n z^{n-1},$$

obtaining

$$\boxed{\begin{aligned} g' &= (zg)' + \frac{1}{1-z}g + 2g + 2h, \qquad g(0) = 1 && (3)\\ h' &= (zh)' + \frac{2}{1-z}h + (zg)' + \frac{1}{1-z}g, \qquad h(0) = 0. && (4) \end{aligned}}$$

Equation (4) is, after a few transformations,

$$\begin{aligned} h' &= \left(\frac{1}{1-z} + \frac{2}{(1-z)^2}\right)h + \frac{z}{1-z}g' + \left(\frac{1}{1-z} + \frac{1}{(1-z)^2}\right)g\\ &= \left(\frac{1}{1-z} + \frac{2}{(1-z)^2}\right)h + \left(-1 + \frac{1}{1-z}\right)g'\\ &\quad + \left(\frac{1}{1-z} + \frac{1}{(1-z)^2}\right)g, \end{aligned} \tag{5}$$

and Eq. (3) becomes

$$g' = \left(\frac{3}{1-z} + \frac{1}{(1-z)^2}\right)g + \frac{2}{1-z}h.$$

Substituting this into Eq. (5), we obtain

$$h' = \left(\frac{1}{1-z} + \frac{2}{(1-z)^2}\right)h + \left(\frac{1}{1-z} + \frac{1}{(1-z)^2}\right)g$$
$$+\left(-1 + \frac{1}{1-z}\right)\left[\left(\frac{3}{1-z} + \frac{1}{(1-z)^2}\right)g + \frac{2}{1-z}h\right],$$

or, upon simplifying,

$$h' = \left(\frac{4}{(1-z)^2} - \frac{1}{1-z}\right)h + \left(\frac{1}{(1-z)^3} + \frac{3}{(1-z)^2} - \frac{2}{1-z}\right)g.$$

Now set

$$g(z) = G\left(\frac{1}{1-z}\right), \qquad \xi = \frac{1}{1-z},$$

and

$$h(z) = H\left(\frac{1}{1-z}\right),$$

obtaining

$$\xi^2 G'(\xi) = (3\xi + \xi^2)G + 2\xi H,$$
$$\xi^2 H'(\xi) = (4\xi^2 - \xi)H + (\xi^3 + 3\xi^2 - 2\xi)G,$$

or

$$\boxed{\begin{aligned} G'(\xi) &= \left(1 + \frac{3}{\xi}\right)G + \frac{2}{\xi}H, \\ H'(\xi) &= \left(4 - \frac{1}{\xi}\right)H + \left(\xi + 3 - \frac{2}{\xi}\right)G. \end{aligned}}$$

Now,

$$\frac{d}{d\xi}(He^{-4\xi}\xi) = (\xi^2 + 3\xi - 2)G(\xi)e^{-4\xi},$$

and, since $H(1) = 0$,

$$H(\xi) = \frac{1}{\xi}e^{4\xi}\int_1^{\xi}(\zeta^2 + 3\zeta - 2)G(\zeta)e^{-4\zeta}\,d\zeta.$$

Thus,

$$G'(\xi) = \left(1 + \frac{3}{\xi}\right)G + \frac{2}{\xi^2}e^{4\xi}\int_1^{\xi}(\zeta^2 + 3\zeta - 2)G(\zeta)e^{-4\zeta}\,d\zeta,$$

or

$$\xi^2 G' e^{-4\xi} = (\xi^2 + 3\xi)Ge^{-4\xi} + 2\int_1^{\xi}(\zeta^2 + 3\zeta - 2)G(\zeta)e^{-4\zeta}\,d\zeta.$$

Now set

$$Ge^{-4\xi} = L(\xi),$$

so that

$$G'(\xi)e^{-4\xi} = 4L(\xi) + L'(\xi),$$

and, hence,

$$\xi^2(L'(\xi) + 4L(\xi)) = (\xi^2 + 3\xi)L(\xi) + 2\int_1^{\xi}L(\zeta)(\zeta^2 + 3\zeta - 2)\,d\zeta.$$

Differentiating with respect to ξ, we have

$$\begin{aligned}&\xi^2 L'' + 4\xi^2 L'(\xi) + 2\xi L'(\xi) + 8\xi L(\xi)\\ &\quad = (2\xi + 3)L(\xi) + (\xi^2 + 3\xi)L'(\xi) + 2(\xi^2 + 3\xi - 2)L(\xi),\end{aligned}$$

or

$$\xi^2 L'' + (3\xi^2 - \xi)L' - (2\xi^2 - 1)L = 0.$$

Finally,

$$\boxed{L''(\xi) + \left(3 - \frac{1}{\xi}\right)L' - \left(2 - \frac{1}{\xi^2}\right)L = 0.}$$

To eliminate the first-order term, we proceed in the routine fashion by setting

$$L(\xi) = M(\xi)Q(\xi),$$

so that

$$L' = M'Q + MQ',$$

and

$$L'' = M''Q + 2M'Q' + MQ''.$$

The differential equation now becomes

$$QM'' + \left[2Q' + \left(3 - \frac{1}{\xi}\right)Q\right]M' + \left[\left(3 - \frac{1}{\xi}\right)Q' + Q'' - \left(2 - \frac{1}{\xi^2}\right)Q\right]M = 0,$$

or

$$M'' + [2Q'/Q + (3 - 1/\xi)]M' + [(3 - 1/\xi)Q'/Q + Q''/Q - (2 - 1/\xi^2)]M = 0.$$

To eliminate the first-order term, set

$$\frac{Q'}{Q} = -\frac{1}{2}\left(3 - \frac{1}{\xi}\right), \qquad Q = e^{-(3/2)\xi}\sqrt{\xi},$$

which, upon differentiation, becomes

$$\frac{QQ'' - Q'^2}{Q^2} = \frac{Q''}{Q} - \left(\frac{Q'}{Q}\right)^2 = \frac{1}{2\xi^2},$$

or

$$\frac{Q''}{Q} = \frac{1}{2\xi^2} + \frac{1}{4}\left(3 - \frac{1}{\xi}\right)^2.$$

The coefficient of M is now

$$\begin{aligned} -\frac{1}{2}\left(3 - \frac{1}{\xi}\right)^2 &+ \frac{1}{4}\left(3 - \frac{1}{\xi}\right)^2 + \frac{1}{2\xi^2} + \frac{1}{\xi^2} - 2 \\ &= -\frac{1}{4}\left(3 - \frac{1}{\xi}\right)^2 + \frac{3}{2}\frac{1}{\xi^2} - 2 \\ &= -\left\{\frac{17}{4} - \frac{3}{2}\frac{1}{\xi} - \frac{5}{4}\frac{1}{\xi^2}\right\}, \end{aligned}$$

and the equation for M is

$$M'' - \left\{\frac{17}{4} - \frac{3}{2}\frac{1}{\xi} - \frac{5}{4}\frac{1}{\xi^2}\right\}M = 0.$$

Thus

$$\boxed{L(\xi) = e^{-(3/2)\xi}\sqrt{\xi}\,M(\xi)}$$

and

$$\boxed{G(\xi) = e^{(5/2)\xi}\sqrt{\xi}\,M(\xi).}$$

For large ξ,

$$G(\xi) \sim \sqrt{\xi}\,e^{(5/2)\xi}e^{(\sqrt{17}/2)\xi} = \sqrt{\xi}\,e^{((5+\sqrt{17})/2)\xi},$$

and then, by steepest descent,

$$\boxed{\log \sigma_n \sim 2\sqrt{\frac{5+\sqrt{17}}{2}}\,\sqrt{n}\,.}$$

3. Other such history-dependent sequences are discussed in a report by Beyer, Schrandt, and Ulam [1].

References

1. W. A. Beyer, R. G. Schrandt, and S. M. Ulam, "Computer Studies of Some History-Dependent Random Processes," LA 4246, Los Alamos Scientific Laboratory, Oct. 28, 1969.